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Spin-glass model with uniform biquadratic interactions

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Abstract. We study a spin-1 infinite-ranged Ising spin-glass model with uniform biquadratic exchange interactions. The phase diagram of the model is obtained in the framework of the replica-symmetric solution. For the case of attractive biquadratic interactions the usual spin-glass phase is stabilized at low temperatures, but for the repulsive case, the antiquadrupolar and antiquadrupolar-glass phases with two-sublattice structure are also possible. We investigate the stability of the replica-symmetric solution and show that the paramagnetic and antiquadrupolar phases are stable, whereas the spin-glass and antiquadrupolar-glass phases are unstable. Parisi's replica-symmetry-breaking procedure is implemented in the neighbourhood of the spin-glass–paramagnetic critical frontier.

1. Introduction

The properties of the infinite-range Ising spin-glass model introduced by Sherrington and Kirkpatrick (SK) [1] have been investigated intensively [2–4]. Such a formulation, which is believed to represent the proper mean-field (MF) approximation for the Ising spin glass, has produced some novel and outstanding results. The rich formalism developed for the SK model has gone far beyond the area of disordered magnetic systems, being employed nowadays in many other complex systems, like neural networks and optimization problems. Even if it happens not to be appropriate for the description of short-range spin glasses, better approximations should consider such a theory as a starting point; indeed, many experimental observations seem to be in good agreement with the predictions of the SK model.

In order to attain a better understanding of the behaviour of real spin glasses, a variety of other infinite-range spin-glass models have been proposed, including Potts, spin-S ($S > 1/2$) and vector spin-glass models [3, 4]. New phases, characterized by different classes of order parameters, have emerged, opening many controversial problems from both the theoretical and experimental points of view.

In this paper we wish to consider an infinite-range spin-1 glass model described by the Hamiltonian

$$\mathcal{H} = -\sum_{(ij)} J_{ij} S_i S_j + D \sum_i S_i^2 - \frac{K}{N} \sum_{(ij)} S_i^2 S_j^2 \tag{1}$$

where $S_i = 0, \pm 1$, the (ij) sums extend over all distinct pairs of spins, D is a crystal field and K, the biquadratic interaction. We assume that the bilinear interactions $\{J_{ij}\}$ are quenched, independent random variables following a Gaussian probability distribution

$$P(J_{ij}) = \left(\frac{N}{2\pi J^2}\right)^{1/2} \exp\left[-\frac{N(J_{ij} - J_0/N)^2}{2J^2}\right]. \tag{2}$$

The presence of N, the total number of spins, in the biquadratic interaction term, as well as in the probability distribution, is necessary to ensure extensivity of thermodynamic quantities. In the limit of no randomness in the bilinear exchange ($J = 0$), this model reduces to the infinite-range version—equivalent to the MF approximation—of the Blume–Emery–Griffiths (BEG) model [5].

Recently, there has been a growing interest in the study of orientational (or quadrupolar) glasses from both experimental and theoretical points of view [6]. Basically, they consist in dilute molecular systems, representing random alloys of interacting quadrupoles, from which the most commonly studied are the mixed alkali cyanides and solid ortho-para-hydrogen mixtures. In the latter systems, the para species (spherically symmetric) play the role of dilution among the ortho (orientable) molecules; for certain ortho-hydrogen concentrations, a low-temperature phase has been observed in which the orientational degrees of freedom of the ortho molecules freeze into a quadrupolar-glass state [7]. Clearly, such systems are expected to present many properties in common with spin glasses, although their theoretical understanding is still far behind. Many quadrupolar-glass models have been proposed as candidates to describe experimental data, most of them with random biquadratic interactions [6]. The model defined through Hamiltonian (1), due to the uniform biquadratic interactions, represents a much simpler theoretical system with axial symmetry, in which quadrupolar-glass behaviour is present.

The model (1) with $K = 0$ and distribution (2) with $J_0 = 0$ has been analysed by various authors [8–12]. As in the case of the MF phase diagram of the BEG model [5], the spin-glass version was shown to exhibit second- and first-order transitions separated by a tricritical point. It is known that the introduction of the biquadratic exchange term has profound effects on the phase diagram of the MF BEG model [13]. In particular, for repulsive biquadratic interactions ($K < 0$), the antiquadrupolar ordering with a two-sublattice structure may be stabilized [14, 15]. In order to describe the two-sublattice phases in the infinite-range model of spin glass, it is necessary to take this possibility into account following the prescription of Korenblit and Shender [16]. Accordingly, we will modify (1) into a two-sublattice Hamiltonian

$$\mathcal{H} = -\sum_{i\in A,\, j\in B} J_{ij} S_i S_j + D \sum_{i\in A,B} S_i^2 - \frac{K}{N} \sum_{i\in A,\, j\in B} S_i^2 S_j^2 \tag{3}$$

where A and B denote the two sublattices with N sites each. We will set up the basic equations for the full Hamiltonian (3) and distribution (2) with $J_0 = 0$, but a detailed discussion will be limited to the case $D = 0$.

This paper is organized as follows. In section 2 we apply the replica method to the two-sublattice Hamiltonian (3) and obtain the stationary equations which describe the system. The replica-symmetric solutions are analysed in section 3. We draw a phase diagram exhibiting four phases namely, the paramagnetic, the antiquadrupolar, spin-glass and antiquadrupolar-glass phases. In section 4 we carry out the stability analysis of the replica-symmetric solution and show its instability throughout both glass phases. It is argued that the critical frontier separating the two glass phases should change within a more appropriate solution. The Parisi solution is implemented in the neighbourhood of the spin-glass–paramagnetic critical frontier and an order-parameter function, similar to that of the SK model, is found. Finally, we present our conclusions in section 6.

2. The replica method

Using the standard replica method [2–4], the free energy per spin f of the system described by Hamiltonian (3) is given by

$$f = -\beta^{-1} \lim_{n\to 0} \frac{1}{n} \left(\lim_{N\to\infty} \frac{1}{2N} \ln \overline{Z^n} \right) \tag{4}$$

where $\beta = 1/k_{\mathrm{B}}T$ and the bar denotes a configurational average. Evaluating $\overline{Z^n}$ for integer n we find

$$\overline{Z^n} = \operatorname{Tr} \exp \left\{ \frac{(\beta J)^2}{N} \sum_{(\alpha\beta)} \left(\sum_{i\in \mathrm{A}} S_i^\alpha S_i^\beta \right) \left(\sum_{j\in \mathrm{B}} S_j^\alpha S_j^\beta \right) - \beta D \sum_\alpha \left[\sum_{i\in \mathrm{A}} (S_i^\alpha)^2 + \sum_{j\in \mathrm{B}} (S_j^\alpha)^2 \right] \right.$$
$$\left. + \frac{\beta}{N} \left(K + \frac{\beta J^2}{2} \right) \sum_\alpha \left[\sum_{i\in \mathrm{A}} (S_i^\alpha)^2 \right] \left[\sum_{j\in \mathrm{B}} (S_j^\alpha)^2 \right] \right\} \tag{5}$$

where $(\alpha\beta)$ denotes all distinct pairs of replicas $(\alpha, \beta = 1, 2, \ldots, n)$. Using the identity

$$\epsilon AB = \tfrac{1}{2}[A^2 + B^2 - (A - \epsilon B)^2] \tag{6}$$

where $\epsilon = \pm 1$, to rewrite the products of sums of spin variables in different sublattices in terms of squares of these quantities and linearizing the resultant expression in the standard way [1], we get

$$\overline{Z^n} = \prod_\alpha \left[\left(\frac{\beta N}{2\pi} \left| K + \frac{\beta J^2}{2} \right| \right)^{3/2} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{A}}^\alpha \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{B}}^\alpha \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}x^\alpha}{\mathrm{i}} \right]$$
$$\times \prod_{(\alpha\beta)} \left[\left(\frac{\beta^2 J^2 N}{2\pi} \right)^{3/2} \int_{-\infty}^{\infty} \mathrm{d}y_{\mathrm{A}}^{\alpha\beta} \int_{-\infty}^{\infty} \mathrm{d}y_{\mathrm{B}}^{\alpha\beta} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}y^{\alpha\beta}}{\mathrm{i}} \right] \exp(-N\phi) \tag{7}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$\phi = \frac{\beta}{2} \left| K + \frac{\beta J^2}{2} \right| \sum_\alpha [(x_{\mathrm{A}}^\alpha)^2 + (x_{\mathrm{B}}^\alpha)^2 - (x^\alpha)^2] + \frac{(\beta J)^2}{2} \sum_{(\alpha\beta)} [(y_{\mathrm{A}}^{\alpha\beta})^2 + (y_{\mathrm{B}}^{\alpha\beta})^2 - (y^{\alpha\beta})^2]$$
$$- \ln \operatorname{Tr} \exp(\overline{\mathcal{H}}_{\mathrm{A}}) - \ln \operatorname{Tr} \exp(\overline{\mathcal{H}}_{\mathrm{B}}). \tag{8}$$

The effective sublattice Hamiltonians $\overline{\mathcal{H}}_{\mathrm{A,B}}$ are given by

$$\overline{\mathcal{H}}_{\mathrm{A}} = \sum_\alpha \left[\beta \left| K + \frac{\beta J^2}{2} \right| (x_{\mathrm{A}}^\alpha - x^\alpha) - \beta D \right] (S^\alpha)^2 + (\beta J)^2 \sum_{(\alpha\beta)} (y_{\mathrm{A}}^{\alpha\beta} - y^{\alpha\beta}) S^\alpha S^\beta \tag{9}$$

$$\overline{\mathcal{H}}_{\mathrm{B}} = \sum_\alpha \left[\beta \left| K + \frac{\beta J^2}{2} \right| (x_{\mathrm{B}}^\alpha + \epsilon x^\alpha) - \beta D \right] (S^\alpha)^2 + (\beta J)^2 \sum_{(\alpha\beta)} (y_{\mathrm{B}}^{\alpha\beta} + y^{\alpha\beta}) S^\alpha S^\beta \tag{10}$$

where $\epsilon = \operatorname{sgn}(K + \beta J^2/2)$.

In the limit $N \to \infty$ we can integrate out the variables x^α and $y^{\alpha\beta}$ using the steepest descent method. The saddle-point equations for these variables are

$$\begin{aligned} x^\alpha &= \langle (S^\alpha)^2 \rangle_{\mathrm{A}} - \epsilon \langle (S^\alpha)^2 \rangle_{\mathrm{B}} \\ y^{\alpha\beta} &= \langle S^\alpha S^\beta \rangle_{\mathrm{A}} - \langle S^\alpha S^\beta \rangle_{\mathrm{B}} \end{aligned} \tag{11}$$

where $\langle \cdots \rangle_{\mathrm{A,B}}$ denote averages with respect to the effective Hamiltonians $\overline{\mathcal{H}}_{\mathrm{A,B}}$. Inserting the saddle-point values of x^α and $y^{\alpha\beta}$ given by (11) into expression (8) for ϕ, the integrations with respect to the variables x_A^α, x_B^α, $y_A^{\alpha\beta}$ and $y_B^{\alpha\beta}$ in equation (7), can be performed using the Laplace method. However, it is more convenient to work with the variables

$$\begin{aligned} p_A^\alpha &= \epsilon(x_B^\alpha + \epsilon x^\alpha) \qquad p_B^\alpha = \epsilon(x_A^\alpha - x^\alpha) \\ q_A^{\alpha\beta} &= y_B^{\alpha\beta} + y^{\alpha\beta} \qquad q_B^{\alpha\beta} = y_A^{\alpha\beta} - y^{\alpha\beta}. \end{aligned} \tag{12}$$

In terms of these new variables the functional ϕ, defined in equation (8), becomes

$$\begin{aligned} \phi = \frac{\beta}{2}\left| K + \frac{\beta J^2}{2}\right| \sum_\alpha [(p_A^\alpha - \langle (S^\alpha)^2\rangle_A) - \epsilon(p_B^\alpha - \langle (S^\alpha)^2\rangle_B)]^2 + \frac{(\beta J)^2}{2} \\ \times \sum_{(\alpha\beta)} [(q_A^{\alpha\beta} - \langle S^\alpha S^\beta\rangle_A) - (q_B^{\alpha\beta} - \langle S^\alpha S^\beta\rangle_B)]^2 + \beta\left(K + \frac{\beta J^2}{2}\right) \\ \times \sum_\alpha p_A^\alpha p_B^\alpha + (\beta J)^2 \sum_{(\alpha\beta)} q_A^{\alpha\beta} q_B^{\alpha\beta} - \ln \mathrm{Tr}\exp(\overline{\mathcal{H}}_A) - \ln \mathrm{Tr}\exp(\overline{\mathcal{H}}_B) \end{aligned} \tag{13}$$

where the effective sublattice Hamiltonians (9) and (10) are given in terms of these new variables by

$$\overline{\mathcal{H}}_{A,B} = \sum_\alpha \beta\left[\left(K + \frac{\beta J^2}{2}\right) p_{B,A}^\alpha - D\right](S^\alpha)^2 + (\beta J)^2 \sum_{(\alpha\beta)} q_{B,A}^{\alpha\beta} S^\alpha S^\beta. \tag{14}$$

In the limit $N \to \infty$ the free energy per spin (4) is obtained according to the Laplace method as

$$f = \frac{k_B T}{2} \lim_{n\to 0} \frac{1}{n} \min[\phi]. \tag{15}$$

The condition for the functional ϕ given by (13) to be stationary is given by the equations

$$p_{A,B}^\alpha = \langle (S^\alpha)^2\rangle_{A,B} \qquad q_{A,B}^{\alpha\beta} = \langle S^\alpha S^\beta\rangle_{A,B.} \tag{16}$$

The elements $p_{A,B}^\alpha$ ($\alpha = 1, 2, \ldots, n$) and $q_{A,B}^{\alpha\beta}$ ($\alpha, \beta = 1, 2, \ldots, n$) in the previous equations, represent the quadrupolar and spin-glass order parameters on each sublattice, respectively. As the result of the biquadratic interactions, the quadrupolar parameters $p_{A,B}^\alpha$ are always non-zero. On the other hand, the spin-glass parameters $q_{A,B}^{\alpha\beta}$, which are zero at high temperatures, may turn up at low temperatures, signalling a spin-glass behaviour. In the next section, we will consider the simplest parametrization for the parameters $p_{A,B}^\alpha$ and $q_{A,B}^{\alpha\beta}$.

3. The replica-symmetric solution

The replica-symmetric solution is obtained by assuming that the order parameters $p_{A,B}^\alpha$ and $q_{A,B}^{\alpha\beta}$ are independent of the replica indices,

$$p_{A,B}^\alpha = p_{A,B} \qquad q_{A,B}^{\alpha\beta} = q_{A,B}. \tag{17}$$

Proceeding in the usual way [1], we find that the stationary conditions (16) become

$$p_{A,B} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-x^2/2)\varphi_2^{A,B}(x) \tag{18}$$

$$q_{A,B} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-x^2/2)[\varphi_1^{A,B}(x)]^2. \tag{19}$$

The functions $\varphi_k^{A,B}(x)$ are defined by

$$\varphi_k^{A,B}(x) = \frac{\mathrm{Tr}\, S^k \exp[\overline{H}_{A,B}(x)]}{\mathrm{Tr} \exp[\overline{H}_{A,B}(x)]} \tag{20}$$

where the trace is taken over the possible spin values ($S = 0, \pm 1$) and the effective sublattice hamiltonians $\overline{H}_{A,B}(x)$ are given by

$$\overline{H}_{A,B}(x) = \beta J \sqrt{q_{B,A}}\, x S + \beta \left[\left(K + \frac{\beta J^2}{2} \right) p_{B,A} - D - \frac{\beta J^2}{2} q_{B,A} \right] S^2. \tag{21}$$

Also the free energy per spin (15) becomes

$$f = \frac{1}{2}\left(K + \frac{\beta J^2}{2} \right) p_A p_B - \frac{\beta J^2}{4} q_A q_B - \frac{1}{2\beta} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \exp(-x^2/2) \ln \mathrm{Tr} \exp[\overline{H}_A(x)]$$
$$- \frac{1}{2\beta} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \exp(-x^2/2) \ln \mathrm{Tr} \exp[\overline{H}_B(x)]. \tag{22}$$

We performed numerical studies of the above equations for the case of zero crystal field ($D = 0$) and found that four types of phases are possible, depending on the temperature T and the biquadratic exchange K:

1. *Paramagnetic* (P): $p_A = p_B$ and $q_A = q_B = 0$.
2. *Antiquadrupolar* (AQ): $p_A \neq p_B$ and $q_A = q_B = 0$.
3. *Spin glass* (SG): $p_A = p_B$ and $q_A = q_B \neq 0$.
4. *Antiquadrupolar glass* (AQG): $p_A \neq p_B$ and $q_A \neq q_B$.

As a typical result, we show in figures 1(a) and 1(b) the possible solutions of equations (18) and (19) found for $K/J = -3.5$. In figure 1(a) we exhibit the possible solutions of the quadrupolar parameters $p_{A,B}$ as a function of the temperature. One notices that the paramagnetic solution is always present, although it is stable only at high temperatures. For intermediate temperatures, the two-sublattice structure with $p_A \neq p_B$ emerges (apart from the presence of the solutions with $p_A = p_B$), representing the stable solutions. Finally, at low temperatures the spin-glass solution with $p_A = p_B$ becomes stable. A similar scenario is observed for the spin-glass parameters $q_{A,B}$ in figure 1(b).

The boundary between the paramagnetic and spin-glass phases is given by the vanishing of the spin-glass order parameter q in the solution $q_A = q_B = q$ and $p_A = p_B = p$ of the equations (18) and (19). The resultant line can be expressed as

$$\frac{K}{J} = -\frac{J}{2k_B T} - \ln\left[2\left(\frac{J}{k_B T} - 1 \right) \right]. \tag{23}$$

The boundary between the paramagnetic and antiquadrupolar phases is determined by the vanishing of the staggered quadrupolar order parameter $p_s = (p_A - p_B)/2$ in the solution $q_A = q_B = 0$, $p_A = p + p_s$ and $p_B = p - p_s$ of (18) and (19). The resultant line is given by

$$\frac{K}{J} = -\frac{1}{p(1-p)} \frac{k_B T}{J} - \frac{J}{2k_B T} \tag{24}$$

where $p = 0.316\,498\ldots$ is the solution of the equation

$$p\left[1 + \frac{1}{2} \exp\left(\frac{1}{1-p} \right) \right] = 1. \tag{25}$$

The two transition lines P–SG and P–AQ, given by (23) and (24), respectively, meet at the multicritical point

$$\frac{k_B T}{J} = 0.316\,498\ldots \qquad \frac{K}{J} = -3.042\,839\ldots. \tag{26}$$

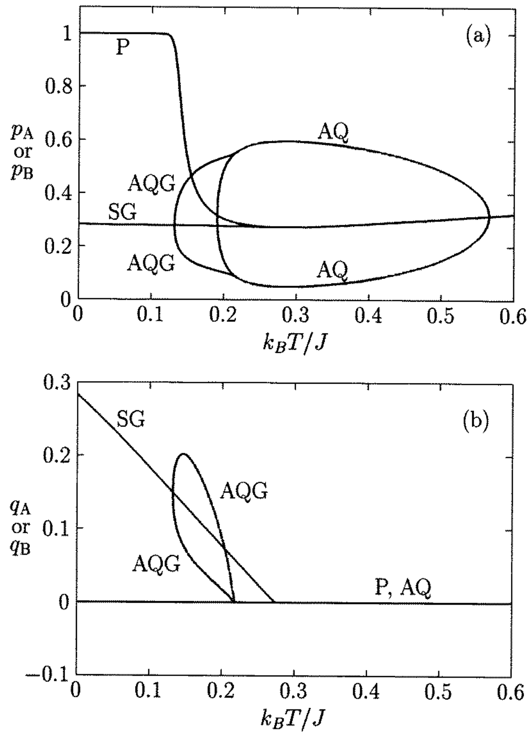

Figure 1. Thermal variation of the possible solutions associated with the (a) quadropolar and (b) spin-glass order parameters for $K/J = -3.5$.

The stability limit of the antiquadrupolar-glass phase is determined by the simultaneous vanishing of the staggered spin-glass order parameter $q_s = (q_A - q_B)/2$ and the staggered quadrupolar order parameter $p_s = (p_A - p_B)/2$ in the solution $q_A = q + q_s$, $q_B = q - q_s$, $p_A = p + p_s$ and $p_B = p - p_s$ of (18) and (19). If $q = (q_A + q_B)/2 = 0$ the transition is to the antiquadrupolar phase, whereas if $q > 0$ the transition is to the spin-glass phase.

The phase diagram within the replica-symmetric solution is exhibited in figure 2. All critical lines correspond to second-order phase transitions; indeed (26) is a tetracritical point, where four second-order critical lines meet. For pure systems, pairs of lines are supposed to meet with the same slope at the tetracritical point. This criterion seems not to hold for spin glasses, being violated in several systems, e.g., the SK model, vector and Potts spin glasses [3, 4]. This is also clearly violated in our phase diagram. For values of K/J to the right of the multicritical point (26), one finds a single phase transition (P–SG) by lowering the temperature. However, for values of K/J to the left of the multicritical point (26), several phase transitions become possible as the temperature decreases. In particular, in the range $-7.5 < K/J < -3.04$ one may get three consecutive phase transitions by lowering the temperature, as one goes through the phases P $\rightarrow$ AQ $\rightarrow$ AQG $\rightarrow$ SG. Usually, phase transitions are associated with a breakdown of symmetry, accompanied by the onset of some new type of order, signalled by order parameters. Therefore, the appearance of an order parameter is commonly connected to a decrease in the entropy of the system. Hence, between the two glass phases, the spin-glass is expected to be the one with the higher entropy. In our phase diagram, one sees that the spin-glass (higher-entropy phase) 'enters' the antiquadrupolar-glass (lower-entropy phase) at low temperatures. As the temperature is diminished, one goes through three phases (P $\rightarrow$ AQ $\rightarrow$ AQG), corresponding to a gradual decrease in the entropy and finally to that for SG, where the entropy increases. This is analogous to what happens in the SK model, for a certain range of J_0/J, within the replica-symmetric solution: one goes from a paramagnetic (high entropy) to a ferromagnetic (low entropy), and then to the SG phase. Although the system does not return to its high-temperature phase, this phenomenon is usually referred to as a 'reentrance' in the spin-glass literature. In that sense, the critical line AQG–SG also corresponds to a 'reentrant' phase transition. In the SK model, this effect is associated with the instability of the replica-symmetric solution at low temperatures; in the next section we will show that such a solution is unstable throughout both AQG and SG phases.

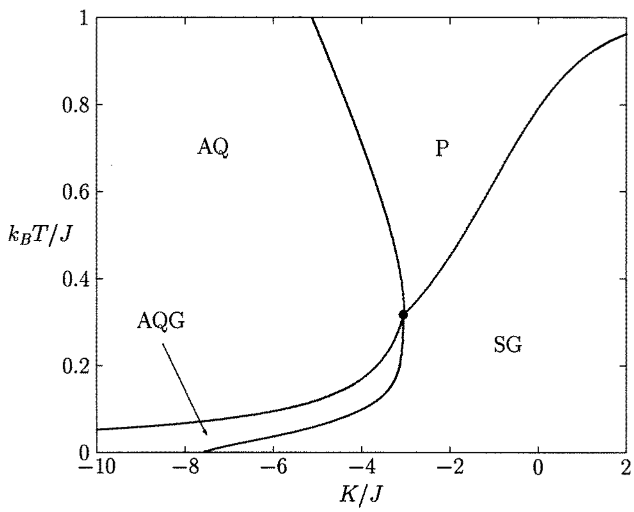

Figure 2. Phase diagram as a function of the temperature T and the biquadratic exchange constant K, showing the paramagnetic (P), antiquadropolar (AQ), spin-glass (SG) and antiquadropolar-glass (AQG) phases.

4. Stability analysis of the replica-symmetric solution

For the application of the Laplace method to be consistent, the replica-symmetric solution (18) and (19) should correspond to the *minimum* of the functional ϕ given by (13). Following de Almeida and Thouless [17], we consider the Hessian matrix of size $n(n+1)$ constructed from the second derivatives of the functional ϕ with respect to the variables $p^{\alpha}_{\mathrm{A,B}}$ and $q^{\alpha\beta}_{\mathrm{A,B}}$. Proceeding in the standard way [17], the eigenvalues of the Hessian matrix can be determined by finding the eigenvectors which divide the space into orthogonal subspaces, closed under replica-index permutations. These eigenvectors are classified into three categories [18]: (a) 4 *longitudinal* eigenvectors independent of replica indices; (b) $4(n-1)$ *anomalous* eigenvectors depending on a single replica index; (c) $n(n-3)$ *transversal* or *replicon* eigenvectors depending on two replica indices. In the limit of $n \rightarrow 0$ the eigenvalues of the longitudinal and anomalous eigenvectors coincide and are given by the eigenvalues of a 4×4 matrix $\boldsymbol{L}$, whereas those associated with the transversal eigenvectors are given by the eigenvalues of a 2×2 matrix $\boldsymbol{T}$.

Let us first consider the eigenvalues associated with the *transversal* eigenvectors. They are the eigenvalues of the matrix $\boldsymbol{T}$ with the elements given by

$$T_{11} = (\beta J)^2(1 + W_{\rm B} + W_{\rm B}^2) \tag{27}$$

$$T_{12} = T_{21} = -(\beta J)^2(W_{\rm A} + W_{\rm B} + W_{\rm A}W_{\rm B}) \tag{28}$$

$$T_{22} = (\beta J)^2(1 + W_{\rm A} + W_{\rm A}^2) \tag{29}$$

where

$$W_{\rm A,B} = (\beta J)^2 \int_{-\infty}^{\infty} \frac{{\rm d}x}{\sqrt{2\pi}} \exp(-x^2/2)\{[\varphi_2^{\rm A,B}(x)]^2 - 2[\varphi_1^{\rm A,B}(x)]^2\varphi_2^{\rm A,B}(x) + [\varphi_1^{\rm A,B}(x)]^4\}. \tag{30}$$

For the replica-symmetric solution to be stable, the eigenvalues of $\boldsymbol{T}$ should be positive. The necessary and sufficient conditions for this are $T_{11} > 0$, $T_{22} > 0$ and $T_{11}T_{22} - T_{12}T_{21} > 0$. The first two conditions are always satisfied, whereas the last condition is equivalent to

$$W_{\rm A}W_{\rm B} < 1. \tag{31}$$

We found numerically that condition (31) is always satisfied in the paramagnetic and antiquadrupolar phases, but violated in the spin-glass and antiquadrupolar-glass phases.

The eigenvalues associated with the longitudinal and anomalous eigenvectors, given by the eigenvalues of the matrix $\boldsymbol{L}$ (see appendix A), were also studied. A numerical analysis of such eigenvalues revealed that they are always real and positive in the paramagnetic and antiquadrupolar phases. However, they may become negative (or even complex, as already observed in similar problems [9, 12]) in the spin-glass and antiquadrupolar-glass phases.

Therefore, the replica-symmetric solution considered in the previous section becomes unstable at low temperatures, throughout both glass phases. From the critical frontiers exhibited in figure 2, P–AQ is inside a stable region, whereas P–SG and AQ–AQG are in the extreme limit for stability of the replica-symmetric solution; such critical lines should not change when one considers solutions in the full replica space. However, the critical frontier AQG–SG is completely inside the unstable region and its location is expected to change within a suitable replica-symmetry-breaking scheme. In the next section we discuss the appropriate low-temperature solution.

5. The Parisi solution

It is a well known fact [2–4] that the instability of the replica-symmetric solution is caused by the assumption that the spin-glass parameters (two-replica indices) are replica-independent; a similar hypothesis for single-replica parameters does not bring any harmful consequences. The parametrization proposed by Parisi [19, 20] for the SK model, although not fully justifiable, is believed to represent the correct solution for such problem [3]. It consists of a hierarchical procedure in which the $n \times n$ spin-glass matrix is broken into blocks according to certain symmetry rules. The same procedure is repeated for the diagonal blocks, and at each step different parameters are introduced. In the $n \to 0$ limit, we are left with an order-parameter function (i.e., an infinite number of order parameters), defined on the interval [0,1]. This scheme has been carried out on other systems, where similar $n \times n$ spin-glass matrices appear, under the same motivation of the one in the SK model, i.e. as an attempt to find a low-temperature stable solution. Due to special symmetry properties, which yield extra terms in the free-energy functional, a single step in the hierarchical process may be enough to ensure stability in some cases, like in the Potts glass [21]. Mostly common however, are spin-glass systems with a replica-symmetry-breaking scheme analogous to the one of the SK model, i.e. a full hierarchical procedure should be followed to attain stability, like m-vector spin glasses [22]. In our case, although this procedure may be implemented throughout both glass phases, we shall restrict its application here for the spin-glass one, in the neighbourhood of its frontier with the paramagnetic phase. In this case, the two-sublattice structure is not necessary and we get the free-energy per spin,

$$f = k_{\mathrm{B}}T \lim_{n\to 0} \frac{1}{n} \min[\phi] \qquad (32)$$

where

$$\phi = \frac{\beta}{2}\left(K + \frac{\beta J^2}{2}\right)\sum_{\alpha}(p^{\alpha})^2 + \frac{(\beta J)^2}{2}\sum_{(\alpha\beta)}(q^{\alpha\beta})^2 - \ln \mathrm{Tr}\exp(\overline{\mathcal{H}}) \qquad (33)$$

$$\overline{\mathcal{H}} = \beta\left(K + \frac{\beta J^2}{2}\right)\sum_{\alpha} p^{\alpha}(S^{\alpha})^2 + (\beta J)^2\sum_{(\alpha\beta)} q^{\alpha\beta}S^{\alpha}S^{\beta}. \qquad (34)$$

Following the preceding discussion, we shall consider the replica-symmetric solution only for the n elements of the quadrupolar parameter, i.e. $p^{\alpha} = p$ ($\alpha = 1, 2, \ldots, n$). Near the critical frontier SG–P, the spin-glass matrix elements $q^{\alpha\beta}$ are small and we may carry out a power-series expansion for the functional ϕ, as shown in appendix B. One sees that the terms in $q^{\alpha\beta}$ are similar to the ones which appear in the corresponding expansion of the SK model [3] (except for the coefficients which assume different values, although preserving the same signs as those of the SK model).

Inspired in the procedure adopted for the SK model, the Parisi prescription [20] may be implemented for the present case, and we get a free-energy functional $f[p, q(x)]$ (see equation (B.3) in appendix B), where $q(x)$ is a function defined in the interval $[0, 1]$. The replica-symmetric solution considered in section 3 is given by $q(x) =$ constant, i.e., $q'(x) = 0$. We are then interested in solutions with $q'(x) \neq 0$, which may be obtained by following the standard procedure, i.e., taking successive derivatives of the equilibrium condition,

$$\frac{1}{q'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{q'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta(\beta f)}{\delta q(x)} = -6A_3x + \mathrm{O}(q_{\mathrm{m}}) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{q'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{q'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta(\beta f)}{\delta q(x)} = -6A_3 + 24(A_4 + B_4x^2)q'(x) + \mathrm{O}(q_{\mathrm{m}}) = 0 \qquad (35)$$

where q_{m} represents the maximum value assumed by the function $q(x)$. The equilibrium condition ($\delta(\beta f)/\delta q(x) = 0$), together with equations (35) yield a function which to lowest order, is given by

$$q(x) = \begin{cases} (A_3/4A_4)x & 0 \leqslant x \leqslant x_1 \\ q_{\mathrm{m}} & x_1 \leqslant x \leqslant 1 \end{cases} \qquad (36)$$

where $x_1 = 4A_2A_4/3A_3^2$ and $q_{\mathrm{m}} = q(1) = A_2/3A_3$. The function $q(x)$ in (36) is similar to that found for the SK model, i.e., a monotonically increasing piece, followed by a plateau. Such an order-parameter function should remove the instability signalled by the negative eigenvalue (although a marginal zero eigenvalue is still expected). However, a complete analysis of the stability of the Parisi solution [23] for the present problem turns out to be a difficult task which is beyond the scope of this paper.

6. Discussion

We have investigated an infinite-range spin-1 glass model in the presence of biquadratic uniform interactions. The model was properly analysed by applying the replica method on a two-sublattice Hamiltonian. The replica-symmetric solution was studied in detail and a phase diagram obtained with four phases, namely: paramagnetic, antiquadrupolar, spin-glass and antiquadrupolar-glass ones. In contrast to the Ghatak–Sherrington spin-1 Ising glass, there is no possibility of first-order transitions. A 'reentrance' effect, with the spin-glass penetrating the antiquadrupolar-glass phase for a certain range of the biquadratic interaction parameter, has been observed. We have performed a stability analysis of the replica-symmetric solution and have shown its instability at low temperatures, i.e., throughout both glass phases. The Parisi replica-symmetry-breaking scheme was considered in the neighbourhood of the spin-glass-paramagnetic critical frontier, and an order-parameter function was obtained, qualitatively similar to that of the Sherrington–Kirkpatrick model. Although possible, the analysis of the Parisi solution valid throughout both glass phases turns out to be a harder task; we believe that within this approach, the reentrance mentioned above should disappear, similar to what happened to the reentrance observed in the phase diagram of the Sherrington–Kirkpatrick model. We speculate that the correct spin-glass–antiquadrupolar-glass critical frontier should be a vertical straight line.

Although we are not aware of any experimental data in agreement with our results, we believe that the present model should be appropriate for the description of quadrupolar glasses with axial symmetry; due to uniform biquadratic interactions, it represents a much simpler model to deal with as compared to those usually employed in the study of such systems.

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Appendix A. The matrix $\boldsymbol{L}$

In the stability analysis of the replica-symmetric solution, the eigenvalues associated with the longitudinal and anomalous eigenvectors coincide in the $n \to 0$ limit. They are given by the eigenvalues of a 4×4 matrix $\boldsymbol{L}$, with elements given by,

$$L_{11} = \beta\left|K + \frac{\beta J^2}{2}\right|(1 + S_{\mathrm{B}} + S_{\mathrm{B}}^2 - 2U_{\mathrm{B}}^2) \tag{A.1}$$

$$L_{22} = \beta\left|K + \frac{\beta J^2}{2}\right|(1 + S_{\mathrm{A}} + S_{\mathrm{A}}^2 - 2U_{\mathrm{A}}^2) \tag{A.2}$$

$$L_{12} = L_{21} = -\epsilon\beta\left|K + \frac{\beta J^2}{2}\right|(S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{A}}S_{\mathrm{B}} - 2\epsilon U_{\mathrm{A}}U_{\mathrm{B}}) \tag{A.3}$$

$$L_{13} = -\frac{1}{2}L_{31} = -\epsilon(\beta J)\left(\beta\left|K + \frac{\beta J^2}{2}\right|\right)^{1/2}(1 + S_{\mathrm{B}} + V_{\mathrm{B}})U_{\mathrm{B}} \tag{A.4}$$

$$L_{24} = -\frac{1}{2}L_{42} = -\epsilon(\beta J)\left(\beta\left|K + \frac{\beta J^2}{2}\right|\right)^{1/2}(1 + S_{\mathrm{A}} + V_{\mathrm{A}})U_{\mathrm{A}} \tag{A.5}$$

$$L_{14} = -\frac{1}{2}L_{41} = (\beta J)\left(\beta\left|K + \frac{\beta J^2}{2}\right|\right)^{1/2}(U_{\mathrm{A}} + \epsilon U_{\mathrm{B}} + S_{\mathrm{B}}U_{\mathrm{A}} + \epsilon U_{\mathrm{B}}V_{\mathrm{A}}) \tag{A.6}$$

$$L_{23} = -\frac{1}{2}L_{32} = (\beta J)\left(\beta\left|K + \frac{\beta J^2}{2}\right|\right)^{1/2}(U_{\mathrm{B}} + \epsilon U_{\mathrm{A}} + S_{\mathrm{A}}U_{\mathrm{B}} + \epsilon U_{\mathrm{A}}V_{\mathrm{B}}) \tag{A.7}$$

$$L_{33} = (\beta J)^2(1 + V_{\mathrm{B}} + V_{\mathrm{B}}^2 - 2U_{\mathrm{B}}^2) \tag{A.8}$$

$$L_{44} = (\beta J)^2(1 + V_A + V_A^2 - 2U_A^2) \tag{A.9}$$

$$L_{34} = L_{43} = -(\beta J)^2(V_A + V_B + V_A V_B - 2\epsilon U_A U_B) \tag{A.10}$$

where

$$S_{A,B} = \beta\left|K + \frac{\beta J^2}{2}\right| \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-x^2/2)\{\varphi_4^{A,B}(x) - [\varphi_2^{A,B}(x)]^2\} \tag{A.11}$$

$$U_{A,B} = (\beta J)\left(\beta\left|K + \frac{\beta J^2}{2}\right|\right)^{1/2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-x^2/2)\{\varphi_3^{A,B}(x)\varphi_1^{A,B}(x) - [\varphi_1^{A,B}(x)]^2\varphi_2^{A,B}(x)\} \tag{A.12}$$

$$V_{A,B} = (\beta J)^2 \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-x^2/2)\{[\varphi_2^{A,B}(x)]^2 - 4[\varphi_1^{A,B}(x)]^2\varphi_2^{A,B}(x) + 3[\varphi_1^{A,B}(x)]^4\}. \tag{A.13}$$

Appendix B. Series expansion for the free-energy functional

In this appendix, we obtain a power-series expansion for the functional ϕ given by equation (33). We shall apply the replica symmetry hypothesis for the quadrupolar elements ($p^\alpha = p, \forall\alpha$), whereas in the neighbourhood of the SG–P critical frontier, the spin-glass matrix elements ($q^{\alpha\beta}$) will be considered as small. One gets the following expansion,

$$\phi(p, q^{\alpha\beta}) = -n \ln D_0 + \tfrac{1}{2} n B p^2 - A_2 \sum_{\alpha\beta} (q^{\alpha\beta})^2 - A_3 \sum_{\alpha\beta\gamma} q^{\alpha\beta} q^{\beta\gamma} q^{\gamma\alpha} - A_4 \sum_{\alpha\beta} (q^{\alpha\beta})^4 - B_4 \sum_{\alpha\beta\gamma\delta} q^{\alpha\beta} q^{\beta\gamma} q^{\gamma\delta} q^{\delta\alpha} + C_4 \sum_{\alpha\beta\gamma} (q^{\alpha\beta})^2 (q^{\beta\gamma})^2 + \cdots \tag{B.1}$$

where the sums over replica labels are now completely unrestricted and the coefficients are given by

$$B = \beta\left(K + \frac{\beta J^2}{2}\right) \tag{B.2a}$$

$$A_2 = \frac{(\beta J)^2}{4}\left[(\beta J)^2\left(\frac{D_2}{D_0}\right)^2 - 1\right] \tag{B.2b}$$

$$A_3 = \frac{(\beta J)^6}{6}\left(\frac{D_2}{D_0}\right)^3 \tag{B.2c}$$

$$A_4 = \frac{(\beta J)^8}{8}\left(\frac{D_2}{D_0}\right)^2\left[\frac{3}{2}\left(\frac{D_2}{D_0}\right)^2 - \frac{D_2}{D_0} + \frac{1}{6}\right] \tag{B.2d}$$

$$B_4 = \frac{(\beta J)^8}{8}\left(\frac{D_2}{D_0}\right)^4 \tag{B.2e}$$

$$C_4 = \frac{(\beta J)^8}{8}\left(\frac{D_2}{D_0}\right)^3\left(3\frac{D_2}{D_0} - 1\right) \tag{B.2f}$$

$$D_0 = 1 + 2\exp(Bp) \tag{B.2g}$$

$$D_2 = 2\exp(Bp). \tag{B.2h}$$

The Parisi prescription [18] may be easily implemented in the expansion (B.1); in the $n \to 0$ limit, we get the free-energy functional,

$$\begin{aligned}\beta f[p, q(x)] = &-\ln D_0 + \frac{B}{2} p^2 + A_2 \langle q^2 \rangle \\ &- A_3 \int_0^1 \mathrm{d}x \left[x q^3(x) + 3q(x) \int_0^x \mathrm{d}y\, q^2(y) \right] + A_4 \langle q^4 \rangle \\ &- C_4 \left\{ \langle q^4 \rangle - 2 \langle q^2 \rangle^2 - \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y [q^2(x) - q^2(y)]^2 \right\} \\ &- B_4 \left\{ \langle q^2 \rangle^2 - 4 \langle q^2 \rangle \langle q \rangle^2 - \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^x \mathrm{d}z \right. \\ &\left. \times [q(x) - q(y)]^2 [q(x) - q(z)]^2 - 4 \langle q \rangle \int_0^1 \mathrm{d}x\, q(x) \int_0^x \mathrm{d}y [q(x) - q(y)]^2 \right\} \\ &+ \cdots \end{aligned} \tag{B.3}$$

where $\langle q^m \rangle = \int_0^1 \mathrm{d}x\, q^m(x)$.

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